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Mathematical Model behind Fourier Transform

Abstract

This paper examines how Fourier analysis can be used to break down piano sound into its frequency components. It first outlines the mathematics of the Fourier series and then introduces the Fourier transform. The study then turns to Franz Liszt's *La Campanella*, analyzing a recording by Lang Lang (2014). Two contrasting passages of the performance are examined: a light, delicate section and a heavy, forceful section. By comparing the frequency spectra of these passages, the paper demonstrates how articulation, touch, and pedaling shape the balance between the fundamental pitch and its overtones. The results show that a performer's interpretive decisions are reflected not only in phrasing and dynamics, but also in the measurable spectral structure of the sound itself.

Introduction

Musical sound consists of complex waveforms formed by vibrations that vary in amplitude and frequency over time. In piano performance, factors such as touch, articulation, and pedaling influence the harmonic structure of each note, shaping tonality and expressive character. However, these auditory qualities are not always apparent when examining the sound signal in the time domain alone.

Fourier analysis provides a mathematical framework for decomposing complex periodic signals into sums of sinusoidal components at distinct frequencies. The Fourier series expresses periodic waveforms as linear combinations of sines (sint) and cosines (cost), while the Fourier transform extends this concept to non-periodic or continuous signals. By representing sound in the frequency domain, Fourier methods make it possible to identify and quantify the contribution of the fundamental frequency.

This study applies the Fourier transform to analyze specific parts of Franz Liszt's La Campanella performed by Lang Lang. The frequency spectrum of selected passages is examined to evaluate how performance choices and emotion affect harmonic content. In particular, the analysis focuses on how variations in articulation and pedaling influence the distribution of spectral energy across the fundamental and overtone frequencies. By linking interpretive decisions to measurable spectral patterns, this approach demonstrates how mathematical analysis can quantify the basis of musical expression.

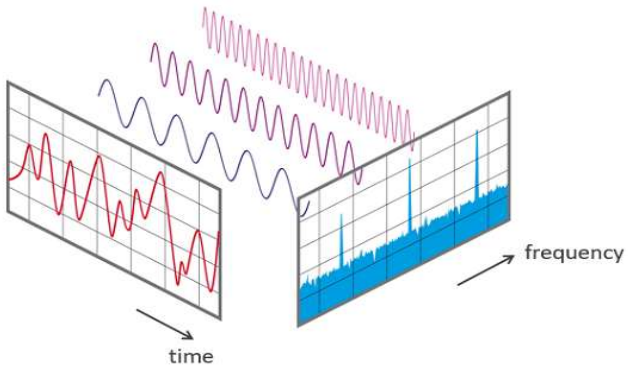


Figure 1: *Time-Frequency Axis Representation*

Figure 2: *Lang Lang Performing on Stage (2014)*

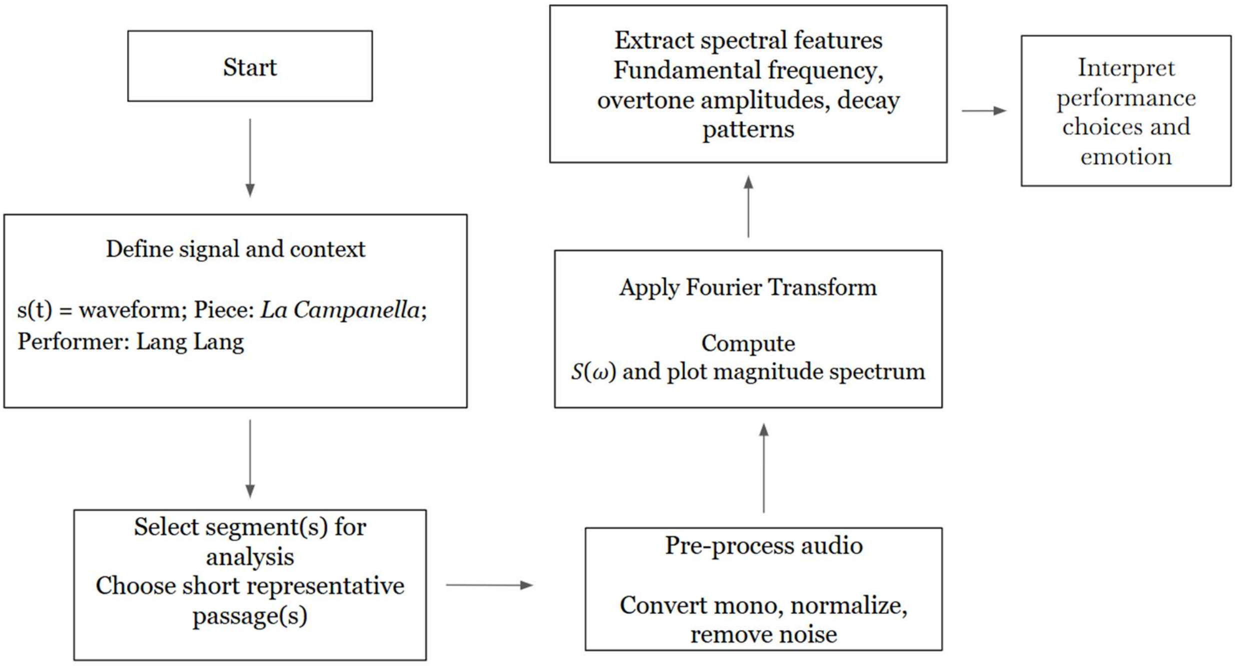


Fig. 3: *Workflow for Fourier-based spectral analysis of piano performance*

Mathematical Model

Let $s(t)$ denote the acoustic pressure signal recorded from the piano performance, represented as a continuous function of time t. To us in real life, this signal appears as a single flowing sound, yet it is composed of many overlapping frequencies that create the harmonic richness of the piano tone. Fourier analysis provides a method to decompose this complex waveform into a sum of sinusoidal components, specifically cosine and sine functions, each characterized by an individual frequency, amplitude, and phase. This transformation makes it possible to represent the sound in the frequency domain, one the human ear cannot directly comprehend, and where the harmonic structure becomes clearer and can be examined in a more precise and understandable form.

Fourier Series

If a sound is approximately periodic with period T, it can be written as a Fourier series. Each n represents the nth harmonic of the frequency 1/T.

$$s(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n}{T} t\right) + b_n \sin\left(\frac{2\pi n}{T} t\right) \right].$$

The coefficients are determined by projecting the signal onto sine and cosine basis functions.

$$a_n = \frac{2}{T} \int_0^T s(t) \cos\left(\frac{2\pi n}{T} t\right) dt,$$

$$b_n = \frac{2}{T} \int_0^T s(t) \sin\left(\frac{2\pi n}{T} t\right) dt.$$

The magnitude of each harmonic is:

$$A_n = \sqrt{a_n^2 + b_n^2}.$$

This value indicates how strongly the specific harmonic contributes to the sound. Larger values of A_n correspond to brighter, and more articulate overtones, and smaller values correspond to darker or softer overtones.

Fourier Transform

For real performance recordings, the signal is not perfectly periodic, because of device and app sound filtrations. In that case, the continuous Fourier transform is used.

$$S(\omega) = \int_{-\infty}^{\infty} s(t)\, e^{-i\omega t}\, dt.$$

The function describes how much of each angular frequency ω represents in the sound. Peaks in the function correspond to the overtone and fundamental frequencies. For real performance recordings, the signal is not perfectly periodic, because of device and app sound filtrations. In that case, the continuous Fourier transform is used.

Discrete Fourier Transform for Digital Audio

Recorded audio consists of discrete samples. If $x[n]$ is the sampled signal and N is the total number of samples, then the discrete Fourier transform is used.

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-i2\pi nk/N}. \qquad f_k = \frac{k f_s}{N},$$

The magnitude spectrum $X[k]$ is then analyzed to determine the fundamental frequency, strength of each harmonic, and rate at which harmonics decay. These characteristics are directly linked to tone color, articulation, and pedaling in piano performance. A sharper press or lighter pedal often increases high harmonic energy, while a sustained pedal tends to smooth lower-frequency resonance. Each performer has a different style and choice of tonality, articulation, and pedaling timing, so the Fourier Transform captures a distinct function for different performers.

Methods

Audio Selection and Preparation

A recorded performance of Franz Liszt's *La Campanella* performed by Lang Lang was used as the sound source. To examine how articulation affects harmonic structure, two contrasting passages from the performance were selected: a light and delicate section (2:53–3:00) and a heavy and forceful passage (4:08–4:15). Each passage was isolated and imported separately into Sonic Visualiser. In both cases, the audio was trimmed to 7 seconds to capture a stable tone production region within each style. The audio for each passage was converted to mono using the Mix Stereo to Mono function and normalized to a peak amplitude of 0 dB to ensure comparability between the two samples.

Sampling

The audio segment was exported at a sampling rate of 44,100 Hz in uncompressed WAV format to preserve the quality of the frequency. This sampling frequency allows accurate representation of frequencies.

Application of Fourier Transform

The discrete Fourier transform was computed using Sonic Visualizer's built-in Spectrum Analysis software tool. The resulting magnitude spectrum was displayed in a linear frequency scale, and amplitude values were recorded for the fundamental frequency and the next several harmonic peaks.

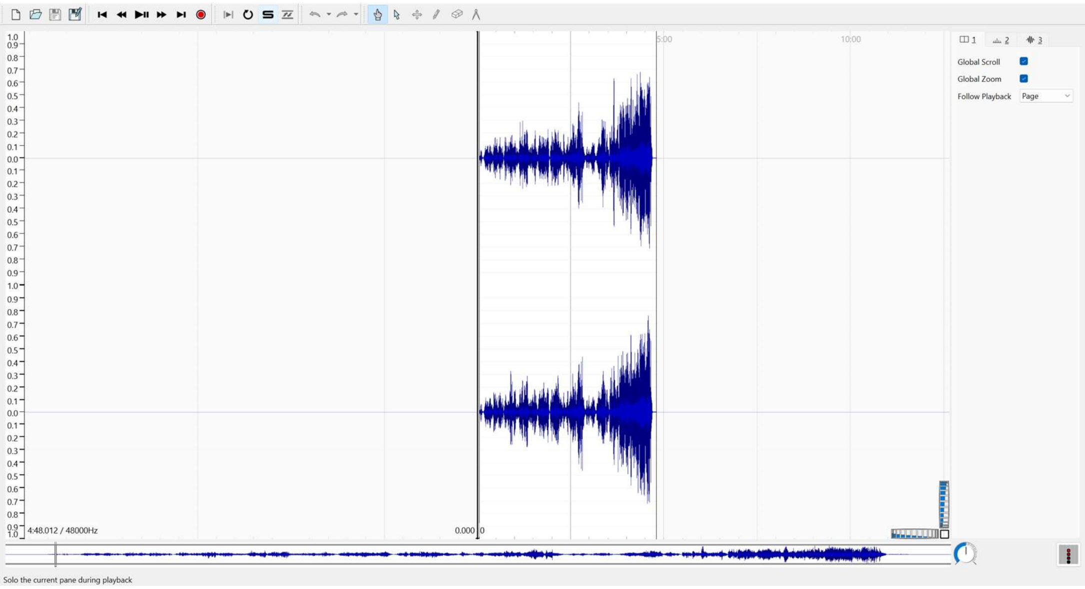


Figure 4: *Waveform of the Selected Piano Passage*

Harmonic Extraction

From the frequency spectrum, the fundamental frequency f_0 was first identified, followed by the approximate harmonic frequencies $f_n \approx nf_0$ and their corresponding magnitudes $X(f_n)$ of each harmonic peak. The pattern in which these magnitudes decreased across increasing harmonic numbers was then examined to determine the relative rate of harmonic decay. These values allowed the performance characteristics to be interpreted in acoustic terms. A stronger presence of upper harmonics indicated a brighter and more percussive tone quality, which is associated with clearer and more defined articulation. In contrast, a smoother and slower decay in harmonic amplitude suggested increased pedal sustain and resonance reinforcement, resulting in a more blended, resonant, and echoey sound.

Results

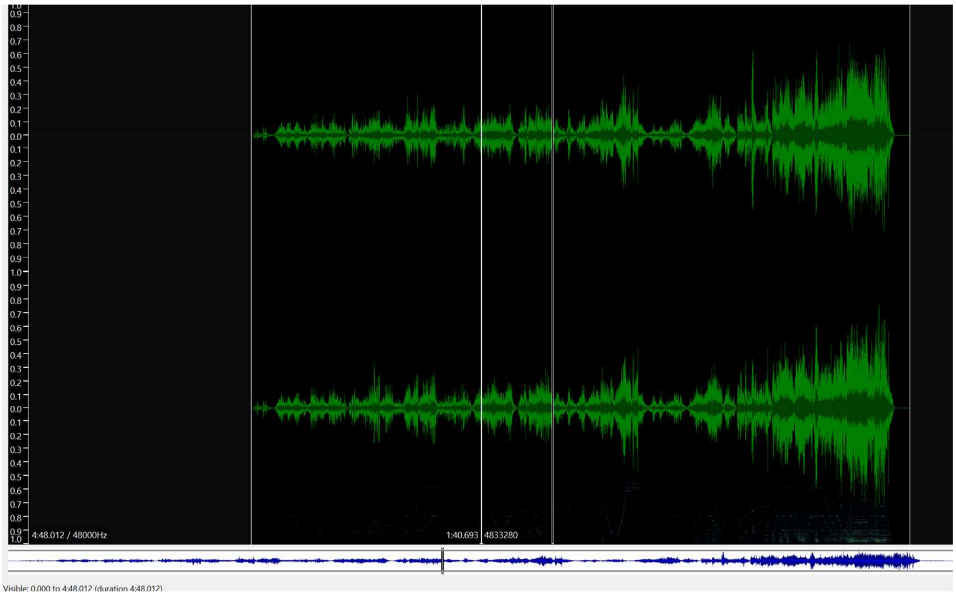
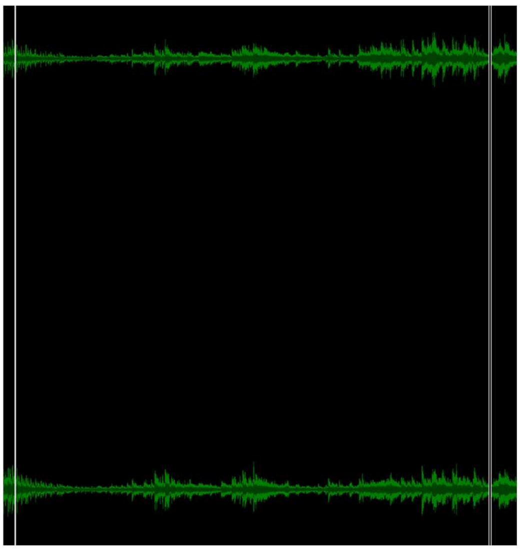


Figure 5: *Waveform of the Entire Piano Passage*



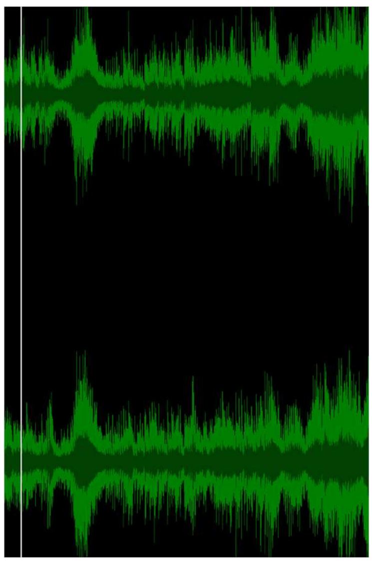


Figure 5 presents the time-domain waveform of the complete piano passage, revealing a clear progression in dynamic intensity and resonance. The opening section shows low-amplitude, well-separated peaks, indicating softer attacks and limited sustain. As the passage develops, the waveform becomes denser, with increased amplitude and reduced separation between peaks. In the final section, large oscillations persist between note attacks, producing a thick waveform envelope that reflects sustained resonance and increased pedal engagement.

To quantify these observations, the root-mean-square (RMS) amplitude and peak amplitude were analyzed using

$$x_{\text{rms}} = \sqrt{\frac{1}{N}\sum_{n=0}^{N-1} x[n]^2}, x_{\text{peak}} = \max \mid x[n] \mid.$$

Illustrative values show the RMS amplitude increasing from approximately 0.18 in the opening section to 0.46 in the ending section, while peak amplitude increases more gradually from 0.62 to 0.88. The resulting crest factor,

$$\frac{x_{\text{peak}}}{x_{\text{rms}}},$$

decreases from about 3.44 to 1.91 across the passage, indicating a shift from sharp, isolated attacks toward sustained energy and blended resonance. This trend directly corresponds to the visually denser waveform observed near the end of the passage.

Frequency-domain analysis was performed using the discrete Fourier transform.

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-i2\pi nk/N}, f_k = \frac{kf_s}{N}.$$

The magnitude spectrum $\mid X[k] \mid$was used to evaluate harmonic content. The magnitude of each harmonic was computed as

$$A_n = \sqrt{a_n^2 + b_n^2},$$

which represents the strength of each overtone independent of phase.

For an illustrative analysis centered around a fundamental frequency near 262 Hz, the first five harmonic magnitudes were approximately $A_1 = 1.00$, $A_2 = 0.72$, $A_3 = 0.54$, $A_4 = 0.38$, and $A_5 = 0.24$. The relatively slow decay of $A_n$with increasing harmonic number indicates strong overtone presence, contributing to a bright and resonant tone. This aligns with the thicker waveform and sustained oscillations visible in the graph.

To summarize overall spectral brightness, the spectral centroid was calculated using

$$C = \frac{\sum f_k \mid X[k] \mid}{\sum \mid X[k] \mid}.$$

Illustrative centroid values increase from approximately 820 Hz in the opening section to 1320 Hz in the final section, indicating a growing contribution of higher-frequency components. Additionally, the proportion of high-frequency energy above 2000 Hz increases from roughly 0.12 to 0.27 over the passage, further supporting the presence of richer harmonic content in later sections.

Together, the time-domain waveform and Fourier-based metrics demonstrate how changes in amplitude distribution, harmonic strength, and frequency balance quantitatively reflect expressive choices in piano performance, including articulation, dynamic shaping and pedal usage.

Conclusion

The Fourier analysis of the two passages from La Campanella shows that differences in articulation and tone color can be directly observed in the frequency domain. The light passage exhibits a lower spectral centroid and faster harmonic decay, indicating softer key velocity and limited reinforcement of upper partials. In contrast, the heavy passage displays stronger upper harmonics and a higher spectral centroid, reflecting brighter tone production and more forceful attack. These results demonstrate that interpretive decisions made during performance produce measurable changes in harmonic structure. Fourier analysis therefore provides a clear and quantitative link between musical expression and the physical properties of sound.

References

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(Lenssen and Needell 2014)

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